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Acoustic scattering at a gap between two orthogonal, semi-infinite barriers: coordinate and spectral equations

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Abstract The 2-D problem of diffraction by a gap between two orthogonal semi-infinite barriers is considered. The method of reflections is applied and the diffraction problem is reformulated as a propagation problem on a multi-sheet surface. An auxiliary problem, with a single incident wave, is formulated. By applying an embedding formula the auxiliary problem is reduced to that of finding a set of edge Green's functions for the surface. The edge Green's functions are proven to satisfy two sets of differential equations: the *coordinate equations* and the *spectral equation*. Although too complicated to be solved analytically, these ordinary differential equations offer some advantages over partial differential equations or integral equations.

Keywords Diffraction · Embedding formulae · Multi-sheet surface · Sommerfeld–Maljuzhinets method · Wiener–Hopf technique

1 Introduction

A classical and well-known application of the Wiener–Hopf technique is the solution of the problem of diffraction by an ideal half-plane [1]. For this problem the directivity diagram of the scattered field is obtained in the neat form of an elementary (algebraic) function. A natural question is, what other diffraction problems can be solved in the same concise and easy manner? In fact, the classical Wiener–Hopf method can be applied only to some half-plane, wedge, and waveguide problems. Over the past 50 years, however, there have been many extensions to the method which enable variations of the technique to be applied to wider range of geometries. For example, one closely related technique, applicable to half-plane and wedge geometries, is the Sommerfeld–Maliuzhinets method [2].

It was Sommerfeld [3] who first noticed that a half-plane diffraction problem can be reformulated as a propagation problem on a branched (two-sheet) surface without scatterers. He also mentioned that some other problems (such as diffraction by a strip or by a grating) could also be reformulated in a similar way. However, no convenient integral representation has been found for those problems. Williams [4] used a similar approach to derive the Green's function for the Helmholtz equation in a wedge, of angle α, $0 < \alpha \leq 2\pi$, with a Dirichlet condition imposed on

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one face and a Neumann condition of the other. He extended the solution domain to 2α and constructed a solution which, when restricted to the physical wedge, satisfied all the appropriate conditions.

This paper is concerned with diffraction problems of the Sommerfeld class, i.e., those that can be reformulated as propagation problems on branched surfaces. A breakthrough in such problems was achieved in works by Latta [5] and Williams [6], who used ordinary differential equations of Fuchsian type to describe the wave field and the directivity diagram. One of the authors of the current paper has continued this work and extended the method of Latta and Williams to the case of diffraction by two strips, [7]. Further studies have led to the technique of coordinate equations, [8]. The essence of this technique is the derivation of a system of two differential equations on the coordinate plane. The coefficients of these equations happen to be elementary (moveover, rational) functions of the coordinates.

In [8] and [9] coordinate equations were derived for the cases of diffraction by several strips on a plane and by a segment on a sphere. Coordinate equations are, however, applicable for any problem on a branched surface. Here a more complicated example is considered, namely the scattering by two ideal half-planes oriented at right angles to each other but not touching. The aim is to develop a general framework, for the coordinate and spectral equations technique, applicable to any problem on a branched surface with branches of order two.

The methods of coordinate and spectral equations are different generalizations of the Wiener–Hopf method. The key feature of the Wiener–Hopf technique is to construct two combinations of functions, one of which is regular in the upper half of the complex plane and the other in the lower half. Each combination contains one (distinct) unknown function which is related to the directivity of the physical problem. If the analytic continuation of each combination is proved to grow no faster than a polynomial, then by Liouville's theorem it is a polynomial. Thus, the unknown functions are determined up to several constants (the coefficients of the polynomial). The derivation of the spectral equation in [7] was similarly achieved by studying various combinations of several unknown functions *and their derivatives*. These combinations were proved to be rational functions, thus ordinary differential equations for the unknown functions were derived. In the framework of the Wiener–Hopf technique, Liouville's theorem is used as a theorem of uniqueness for analytic functions in the complex plane. The idea of the coordinate equations method is to use the uniqueness of wave fields on the physical coordinate plane.

The paper is organized as follows. In Sect. 2 the initial physical problem of scattering at a gap between two orthogonal semi-infinite barriers is stated. The method of reflections is applied in order to reformulate this as a propagation problem on a surface having four sheets and four finite branch points. An auxiliary propagation problem with a single incident plane wave is then formulated. In Sect. 3 two embedding formulae are derived, the second of which expresses the directivity diagram of the physical field as a combination of the directivities of the edge Green's functions. The edge Green's functions are wave fields due to dipole sources located at the branch points. In Sect. 4 the coordinate equations are derived for the edge Green's functions and, using these, a spectral equation is derived in Sect. 5. Numerical perspectives of the method are considered briefly in Sect. 6 and some conclusions are presented in Sect. 7.

2 Diffraction problem on the physical plane and on the branched surface

2.1 Formulation of the physical problem

Consider the problem of scattering by two half-planes. We assume that the problem is 2-D in spatial coordinates. The lines γ_1 and γ_2 in Fig. 1 are the cross-sections of the half-planes cut by the (x, y)-plane.

Assume that the problem is stationary, i.e., the time dependence has form of $\mathrm{e}^{-\mathrm{i}\omega t}$ and it is omitted henceforth. The Helmholtz equation,

$$\Delta u + k_0^2 u = 0, \tag{1}$$

where $u(x, y)$ is the fluid velocity potential, is fulfilled everywhere on the (x, y)-plane, except for the scatterers γ_1 and γ_2.

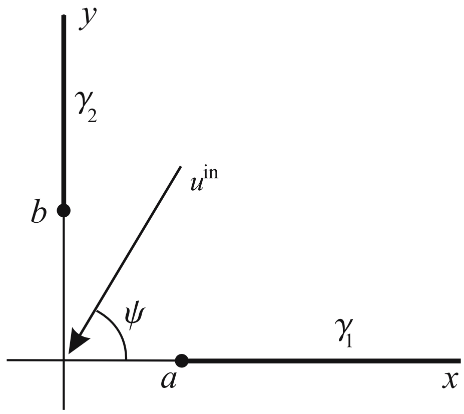

Fig. 1 Geometry of the problem

The boundary conditions are as follows: on both sides of γ_1 and γ_2 the normal derivative of the field is equal to zero. That is,

$$\partial_n u = 0. \tag{2}$$

In the vicinities of the points $(a, 0)$ and $(0, b)$ one must impose Meixner's conditions. In the local cylindrical coordinates ρ, ϕ near each of the points the asymptotics of the field must have the form of Meixner's series

$$u = c\rho^{1/2} \cos \frac{\phi}{2} + O(\rho^{3/2}) + \text{regular terms},$$

where the coefficient c depends on k and on the incidence angle. Note that the coordinate ϕ is equal to 0 and 2π on the surfaces of the corresponding scatterer.

The incident wave has form of a plane wave,

$$u^{\text{in}} = \exp\{-\mathrm{i}k_0(x \cos \psi + y \sin \psi)\}, \tag{3}$$

where ψ is the angle of incidence. Physically, the cases $0 < \psi < \pi/2$ and $\pi/2 < \psi < 2\pi$ are very different, since the incident wave travels either in the interior of the angle or in its exterior. Mathematically, however, these cases are close to each other.

One must also impose a radiation condition. We assume that there are no wave components coming from infinity except u^{in}, i.e., the total field is the sum of an incident field, a reflected field and a scattered field. The last one has the form of an outgoing cylindrical wave far from the origin.

2.2 Reformulation of the diffraction problem

A multi-sheet surface can be constructed using the principle of reflection. Make two cuts along the lines γ_1 and γ_2 in the plane (x, y). Let the field $u(x, y)$ be a solution of the diffraction problem formulated in the previous subsection. This field is defined on the cut plane, which we will henceforth refer to as the *physical sheet*.

Three new sheets can be created by reflecting the physical sheet with respect to the coordinate axes. Since, in the physical problem, the lines γ_1 and γ_2 are described by the Neumann condition, the new sheets bear the wave fields $u(-x, y)$, $u(-x, -y)$ and $u(x, -y)$.[1] A branched surface can now be assembled by connecting the four sheets (the

[1] Dirichlet conditions can also be accommodated in this way, but the waves fields would then be $-u(x, -y)$, $u(-x, -y)$, $-u(-x, y)$.

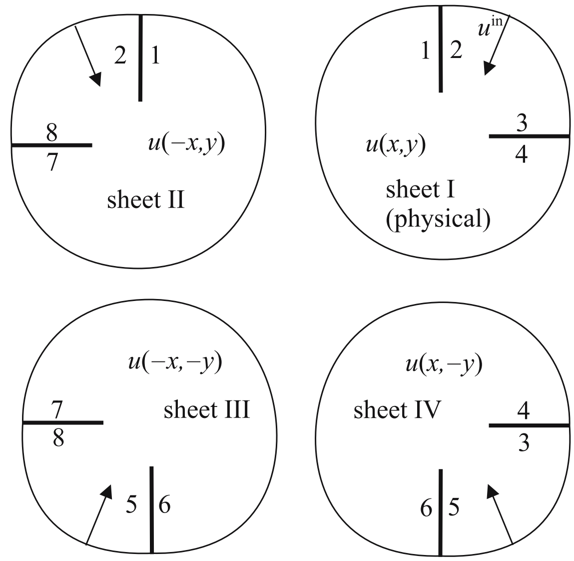

Fig. 2 Topology of the branched surface

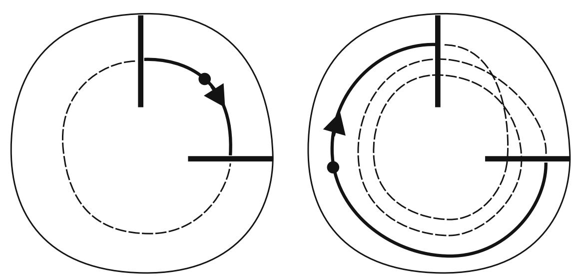

Fig. 3 Contours encircling infinity

physical one and three reflected ones) according to the scheme sketched in Fig. 2. The shores marked by the same numbers should be attached to each other.

As a result, the wave field is now defined on a branched surface having four sheets and four finite branch points. Each branch point has order two, i.e., a bypass encircling a branch point twice ends where it starts. The finite branch points have coordinates $(a, 0)$, $(0, b)$, $(-a, 0)$, and $(0, -b)$, and these points are henceforth referred to as the *vertices* of the surface. Note that infinity is also a branch point of the surface. The topology of the branched surface at infinity can be studied by drawing contours encircling infinity, as shown in Fig. 3. If the contour starts at the area $0 < \phi < \pi/2$ of the physical sheet, i.e., at the inner part of the scatterer, then it ends at the same point after a single bypass. If the contour starts at the area $\pi/2 < \phi < 2\pi$, i.e., at the outer part of the scatterer, then it should make three bypasses before reaching the physical sheet again (see Fig. 3). Thus, infinity is a branch point of order 3. In the vicinity of this point the surface can be separated into four sheets. The first sheet is isolated at infinity, and three other sheets are connected.

One can easily show that the field is continuous everywhere on the surface, and that its spatial derivatives are continuous everywhere, except at the branch points. This means that the field satisfies the Helmholtz equation (1) everywhere, except at the branch points.

A diffraction problem can now be formulated on the branched surface described above. Let the field on the surface obey Helmholtz's equation everywhere except at the vertices. Further, let the field obey Meixner's conditions at the vertices. Thus, the local asymptotics near each vertex have form

$$u = C_1 \rho^{1/2} \cos\frac{\phi}{2} + C_2 \rho^{1/2} \sin\frac{\phi}{2} + O(\rho^{3/2}) + \text{regular terms} \tag{4}$$

in local polar coordinates, where *regular terms* denotes a function which is single-valued and analytical in (x, y)-coordinates. The incident field for this problem comprises four plane waves travelling in four symmetrical directions along different sheets of the surface (the directions are marked in Fig. 2).

To define a radiation condition, consider a field generated by a point source located on one of the sheets of the surface. The wave field should have form of an outgoing cylindrical wave,

$$u(R, \varphi, N) = \frac{\mathrm{e}^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}} D(\varphi, N) + O(R^{-3/2}\mathrm{e}^{\mathrm{i}k_0 R}), \tag{5}$$

far from the origin. Here (R, φ) are the global polar coordinates defined by the relations $x = R\cos\varphi$, $y = R\sin\varphi$; N is the index of the sheet (according to Fig. 2) and D is the directivity diagram of the field.

An auxiliary propagation problem may be formulated on the branched surface. Instead of four incident plane waves coming from the directions $\pm\psi$, $\pm\psi + \pi$, consider a problem with a single incident plane wave. Thus, let $u(\psi, M; x, y, N)$ denote the field on the Nth sheet, given that the incident wave is propagating at angle ψ on the Mth sheet. For convenience we henceforth consider the incident wave to be coming from the direction ψ along the physical sheet. Thus, the solution of the auxiliary propagation problem is denoted by $u(\psi, 1; x, y, N)$. Note that the solution of the auxiliary problem does not correspond to any physical diffraction problem. The solution of the initial diffraction problem on the physical sheet can, however, be obtained from the auxiliary field by a symmetrization:

$$u(x, y) = u(\psi, 1; x, y, 1) + u(\psi, 1; -x, y, 2) + u(\psi, 1; -x, -y, 3) + u(\psi, 1; x, -y, 4). \tag{6}$$

Note that, a plane incident wave is the limit of a point-source wave as the distance to the source tends to infinity while the amplitude of the source changes in an appropriate way. Thus, formally, the auxiliary field can be considered as a field due to a distant point source. The source should be located on the physical sheet and should have a proper amplitude, i.e., the auxiliary field should be a solution of the inhomogeneous Helmholtz equation

$$\Delta u + k_0^2 u = F(R_0) \frac{\delta(R - R_0)\delta(\varphi - \psi)\delta_{N,1}}{R_0}, \qquad F(R_0) = \frac{4\sqrt{\pi k_0 R_0}\,\mathrm{e}^{-\mathrm{i}k_0 R_0 + 3\pi\mathrm{i}/4}}{\sqrt{2}}, \tag{7}$$

taken as $R_0 \to 0$. Here $\delta(.)$ is the Dirac delta function, δ_{NM} is the Kronecker delta, and R_0 and ψ are cylindrical coordinates of the source. The factor $F(R_0)$ is chosen such that the field of the point source behaves near the origin as a plane wave of a unit amplitude. Equation (7) should be solved taking into account the Meixner and radiation conditions.

In what follows we concentrate our efforts on determining the function $u(\psi, 1; x, y, N)$. Consider the field $u(\psi, M; x, y, N)$ far from the origin. There are optically illuminated zones where the incident plane wave is observable, shadow zones, and penumbral zones connecting illuminating and shadow areas. The behaviour of the field depends on the sheet (considered at infinity), along which the incident plane wave is coming to the vertices. If the plane wave comes from the sheet isolated at infinity, then all this sheet is fully illuminated, while three connected sheets are in the shadow and there are no penumbral areas. If, however, the incident wave comes along one of the sheets branched at infinity, then some parts of the branched sheets are illuminated, some are not illuminated, and there are two penumbral zones. The isolated sheet is in the shadow in this case.

2.3 "Complex" variables

It is convenient to introduce new variables z and $\bar{z}$ related to x and y by the formulae

$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y. \tag{8}$$

The Helmholtz equation can then be written in complex coordinates as:

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{k_0^2}{4} u = 0. \tag{9}$$

The coordinates of the vertices can also be expressed in complex form:

$$p_1 = a, \quad p_2 = \mathrm{i}b, \quad p_3 = -a, \quad p_4 = -\mathrm{i}b. \tag{10}$$

Then, near the vertices, local complex coordinates can be defined by:

$$z_{m+} = z - p_m, \quad z_{m-} = \bar{z} - \bar{p}_m, \quad m = 1 \ldots 4. \tag{11}$$

3 Embedding formula

Embedding formulae were introduced by Williams [6]. An embedding formula can be a convenient way of representing the dependence of the directivity diagram on the angle of incidence. The idea of embedding is to substitute an arbitrary plane-wave incidence problem by a set (a basis) of auxiliary functions. This technique was also successfully used by Martin and Wickham [10]. In this article we use the approach described in [11] to derive the embedding formulae. For the sake of brevity only an outline of the procedure is presented; further details can be found in that paper.

It is important to note that, henceforth, we assume that the theorem of uniqueness is valid on the surface, i.e., if, for given k_0, function u obeys the Helmholtz equation (1) on the surface, the radiation condition at infinity and the Meixner conditions at the vertices, then u is equal to zero everywhere on the surface.

3.1 Edge Green's functions on the branched surface

Introduce local cylindrical coordinates (ρ_m, ϕ_m), $m = 1 \ldots 4$ near each vertex of the branched surface as shown in Fig. 4. Each angle ϕ_m varies from 0 to 4π. The local coordinates describe only the vicinities of the branch points, i.e., they are not used to describe the sheets that are not branched.

Consider an inhomogeneous Helmholtz equation

$$\Delta u + k_0^2 u = S_{m\pm}, \tag{12}$$

where the sources are given by the formulae

$$S_{m\pm} = \frac{\pi^{1/2}}{\epsilon^{1/2}} \delta(\rho_m - \epsilon)[\delta(\phi_m) \mp \mathrm{i}\delta(\phi_m - \pi) - \delta(\phi_m - 2\pi) \pm \mathrm{i}\delta(\phi_m - 3\pi)], \tag{13}$$

and ϵ is a small positive value.

There are eight sources, two per vertex. For each source it is necessary to find the solution of the propagation problem defined by (12) together with the radiation condition and Meixner's conditions at the vertices. On taking

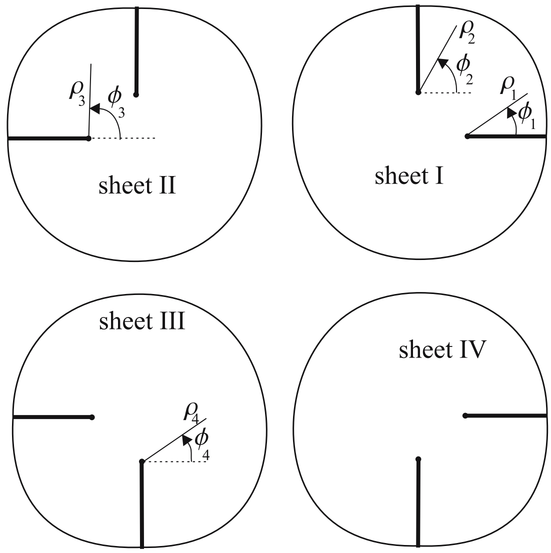

Fig. 4 Local coordinates near branch points

the limit $\epsilon \to 0$, eight solutions $u^{m\pm}$ are obtained. These will henceforth be referred to as *the edge Green's functions* of the surface. Each of the edge Green's functions depends on the spatial coordinates (x, y) and on the index of the sheet of the surface N, i.e., $u^{m\pm} = u^{m\pm}(x, y, N)$.

Although each solution obeys the Meixner conditions for $\epsilon \neq 0$, the limits do not obey these conditions. An elementary analysis shows that the local asymptotics of the solutions have form of

$$u^{m+}(z_{n+}, z_{n-}) = -\frac{\delta_{m,n}}{\sqrt{\pi}} z_{n+}^{-1/2} + \frac{2}{\sqrt{\pi}}\left(C^{m+}_{n+} z_{n+}^{1/2} + C^{m+}_{n-} z_{n-}^{1/2}\right) + O(\rho_n^{3/2}) + \text{regular terms}, \tag{14}$$

$$u^{m-}(z_{n+}, z_{n-}) = -\frac{\delta_{m,n}}{\sqrt{\pi}} z_{n-}^{-1/2} + \frac{2}{\sqrt{\pi}}\left(C^{m-}_{n+} z_{n+}^{1/2} + C^{m-}_{n-} z_{n-}^{1/2}\right) + O(\rho_n^{3/2}) + \text{regular terms}, \tag{15}$$

where $C^{m\pm}_{n\pm}$ are coefficients.

Two branches of the square roots in (14), (15) correspond to two sheets of the branched surface that are connected at the branch point, i.e., the asymptotics (14), (15) are valid only on two sheets of the surface, not on all of them. Henceforth, two arguments (z_{n+}, z_{n-}) will be used instead of three arguments $(z, \bar{z}, N)$ to describe the local behaviour of a solution near one of the vertices. One can see that the first terms in (14), (15) are *over-Meixner*, i.e., they are prohibited by Meixner's conditions.

3.2 "Weak" Embedding formula

Theorem 1 *Let the function $u(\psi, 1; x, y, N)$ have the following local asymptotics*:

$$u(\psi, 1; z_{n+}, z_{n-}) = \frac{2}{\sqrt{\pi}}\left(C_{n+} z_{n+}^{1/2} + C_{n-} z_{n-}^{1/2}\right) + O(\rho_n^{3/2}) + \textit{regular terms} \tag{16}$$

for some coefficients $C_{n\pm}$. Let H_x be the operator

$$H_x = \partial_x + \mathrm{i}k_0 \cos\psi = \partial_z + \partial_{\bar{z}} + \mathrm{i}k_0 \cos\psi. \tag{17}$$

Then

$$H_x[u(\psi, 1; x, y, N)] = -\sum_{m=1}^{4} \left[C_{m+} u^{m+}(x, y, N) + C_{m-} u^{m-}(x, y, N)\right]. \tag{18}$$

Relation (18) is the weak embedding formula. To prove this theorem consider the quantity $H_x[u(\psi, 1; x, y, N)]$. It obeys the homogeneous Helmholtz equation everywhere except at the vertices, since the Helmholtz operator commutes with translations. Moreover, $H_x[u(\psi, 1; x, y, N)]$ obeys the radiation condition, since the operator H_x nullifies the incident plane wave.

Consider the local asymptotics of $H_x[u(\psi, 1; x, y, N)]$ at the vertices. Due to differentiation, the asymptotics contain the following over-Meixner terms:

$$H_x[u(\psi, 1; z_{n+}, z_{n-})] = \frac{C_{n+} z_{n+}^{-1/2} + C_{n-} z_{n-}^{-1/2}}{\sqrt{\pi}} + \text{Meixner terms}.$$

Now, construct the function

$$w = H_x[u(\psi, 1; x, y, N)] + \sum_{m=1}^{4} \left[C_{m+} u^{m+}(x, y, N) + C_{m-} u^{m-}(x, y, N)\right]. \tag{19}$$

This function obeys Helmholtz's equation, the radiation condition, and Meixner's conditions at the vertices. Therefore it should be identically equal to zero due to our assumption of the uniqueness of the solution.

3.3 "Strong" embedding formula

Formula (18) has two significant disadvantages: firstly, it contains unknown constants $C_{m\pm}$; secondly, the field u must be reconstructed from (18) by solving an ordinary differential equation with respect to the variable x. These issues provide an incentive for strengthening result (18).

In the illuminated zones the field can be represented as

$$u(\psi, 1; x, y, N) = u^{\text{in}} + u^{\text{sc}} + O(R^{-3/2} \mathrm{e}^{\mathrm{i}k_0 R}),$$

whilst in the shadow zones it can be expressed as

$$u(\psi, 1; x, y, N) = u^{\text{sc}} + O(R^{-3/2} \mathrm{e}^{\mathrm{i}k_0 R}),$$

where u^{sc} is a cylindrical scattered wave

$$u^{\text{sc}}(\psi, 1; R, \varphi, N) = \frac{\mathrm{e}^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}} D(\psi, 1; \varphi, N), \tag{20}$$

and D is the directivity.

The directivities of the edge Green's functions are denoted by $U^{m\pm}(\varphi, N)$, thus:

$$u^{m\pm}(R, \varphi, N) = \frac{\mathrm{e}^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}} U^{m\pm}(\varphi, N) + O(R^{-3/2} \mathrm{e}^{\mathrm{i}k_0 R}). \tag{21}$$

Theorem 2 *The directivity of the field $u(\psi, 1; x, y, N)$ is connected with the directivities of the edge Green's functions by the formula*

$$D(\psi, 1; \varphi, N) = -\frac{\sum_{m=1}^{4} [U^{m+}(\psi, 1) U^{m+}(\varphi, N) + U^{m-}(\psi, 1) U^{m-}(\varphi, N)]}{4k_0(\cos\varphi + \cos\psi)}, \tag{22}$$

Relation (22) is the strong embedding formula. The proof of this theorem closely follows that given by Craster et al. [11]. The first step is to prove the important relation

$$C_{m\pm} = \frac{\mathrm{i}}{4} U^{m\pm}(\psi, 1), \tag{23}$$

where $C_{n\pm}$ are defined in (18). Note that $C_{n\pm}$ can be calculated using the following expression

$$C_{n\pm} = \frac{1}{8} \lim_{\epsilon \to 0} \iint u(\psi, 1; x, y, N) S_{n\pm} \mathrm{d}s, \tag{24}$$

where $S_{n\pm}$ are given by (13), and the integration is performed over the whole surface. Note also that in (24) the field $u(\psi, 1; x, y, N)$ is multiplied by the sources of the field $u^{n\pm}(x, y, N)$. According to the reciprocity principle, one can replace this integral by another one, in which the sources of the field $u(\psi, 1; x, y, N)$ are multiplied by the field $u^{n\pm}(x, y, N)$, thus

$$C_{n\pm} = \lim_{R_0 \to \infty} \lim_{\epsilon \to 0} \frac{1}{8} \iint \frac{\mathrm{e}^{\mathrm{i}k_0 R}}{\sqrt{2\pi k_0 R}} U^{n\pm}(\varphi, 1) \frac{F(R_0)}{R_0} \delta(R - R_0) \delta(\varphi - \psi) \mathrm{d}s, \tag{25}$$

where $F(R_0)$ is taken from (7). The limit leads to (23).

The second step is to show that the operator H_x acts on a directivity according to the relation:

$$D(\varphi, N) \xrightarrow{H_x} \mathrm{i}k_0(\cos\varphi + \cos\psi) D(\varphi, N). \tag{26}$$

This relation can be proved as follows. If the total field contains no incident field, reflected field and penumbral zones then it is described by the asymptotics of the form (5), and these asymptotics can be differentiated with respect to the coordinates (The last fact can be proved by applying Green's theorem). Then a direct application of the operator H_x to the asymptotics gives (26). If, on the other hand, the total field contains incident and reflected waves, then one can subtract a perfect wedge solution from the solution of the problem having the wedge with a gap as a scatterer. The difference will contain no incident or reflected waves, so the relation (26) should be valid for it. For the perfect wedge, relation (26) can be checked directly. On applying (26) to (18) and using (23), one obtains Eq. 22.

In this section two embedding formulae have been derived. These reformulate the physical problem in terms of the, as yet unknown, edge Green's functions. Some results pertaining to the edge Green's functions will be derived in the next two sections.

4 The coordinate equations

4.1 Derivation of the coordinate equations

Consider the vector of unknowns

$$\mathbf{u}(x, y) = (u^{1+}, u^{2+}, u^{3+}, u^{4+}, u^{1-}, u^{2-}, u^{3-}, u^{4-})^T \tag{27}$$

composed of the edge Green's functions and introduce the following notation: let $\mathbf{I}$ be the 4×4 identity matrix; let $\mathbf{P} = \mathrm{diag}(p_1, p_2, p_3, p_3)$, and $\bar{\mathbf{P}}$ be the complex conjugate of $\mathbf{P}$; let $\mathbf{C}_-^+$ and $\mathbf{C}_+^-$ be 4×4 matrices with elements C_{n-}^{m+} and C_{n+}^{m-}, respectively (m is row, n is column); and finally let $\hat{\mathbf{C}}_+^+$ and $\hat{\mathbf{C}}_-^-$ be matrices with elements

$$\hat{C}_{n+}^{m+} = (p_n - p_m) C_{n+}^{m+}, \qquad \hat{C}_{n-}^{m-} = (\bar{p}_n - \bar{p}_m) C_{n-}^{m-}.$$

Theorem 3 *Vector* $\mathbf{u}$ *obeys the system of differential equations*

$$\frac{\partial}{\partial z}\mathbf{u} = \mathbf{Z}_+ \mathbf{u}, \qquad \frac{\partial}{\partial \bar{z}}\mathbf{u} = \mathbf{Z}_- \mathbf{u}, \tag{28}$$

where the coefficients $\mathbf{Z}_+$ *and* $\mathbf{Z}_-$ *are given in block form by the formulae*

$$\mathbf{Z}_+ = -\begin{pmatrix} (z\mathbf{I} - \mathbf{P})^{-1} \cdot (\mathbf{I}/2 + \hat{\mathbf{C}}_+^+), & (z\mathbf{I} - \mathbf{P})^{-1} \cdot \mathbf{C}_-^+ \cdot (\bar{z}\mathbf{I} - \bar{\mathbf{P}}) \\ \mathbf{C}_+^-, & 0 \end{pmatrix}, \tag{29}$$

$$\mathbf{Z}_- = -\begin{pmatrix} 0, & \mathbf{C}_-^+ \\ (\bar{z}\mathbf{I}-\bar{\mathbf{P}})^{-1}\cdot\mathbf{C}_+^-\cdot(z\mathbf{I}-\mathbf{P}), & (\bar{z}\mathbf{I}-\bar{\mathbf{P}})^{-1}\cdot(\mathbf{I}/2+\hat{\mathbf{C}}_-^-), \end{pmatrix}. \tag{30}$$

Equations 28 with coefficients (29), (30) are the *coordinate equations* for the problem.

The proof of Theorem 3 is similar to the that of Theorem 1 in that several combinations of the unknown functions and their derivatives are constructed. These quantities obey the conditions of the theorem of uniqueness, namely they obey the Helmholtz equation, the radiation condition and Meixner's conditions, and thus are equal to zero.

First, consider the following derivatives:

$$\frac{\partial u^{m+}}{\partial \bar{z}}, \qquad \frac{\partial u^{m-}}{\partial z}, \qquad \frac{\partial u^{m\pm}}{\partial \phi_m} = \mathrm{i}\left((z-p_m)\frac{\partial}{\partial z} - (\bar{z}-\bar{p}_m)\frac{\partial}{\partial \bar{z}}\right)u^{m\pm},$$

where $m = 1\ldots 4$. Obviously each of these obeys the Helmholtz equation and the radiation condition, and they have local singularities no higher than $\rho_m^{-1/2}$. On compensating the vertex singularities in these derivatives by adding appropriate combinations of the edge Green's functions and using the uniqueness theorem, it is found that

$$\frac{\partial u^{m+}}{\partial \bar{z}} = -\sum_{n=1}^{4} C_{n-}^{m+}u^{n-}, \quad \frac{\partial u^{m-}}{\partial z} = -\sum_{n=1}^{4} C_{n+}^{m-}u^{n+}, \tag{31}$$

$$\frac{\partial u^{m\pm}}{\partial \phi_m} = \mp\frac{\mathrm{i}}{2}u^{m\pm} - \mathrm{i}\sum_{n=1}^{4}[(p_n-p_m)C_{n+}^{m\pm}u^{n+} - (\bar{p}_n-\bar{p}_m)C_{n-}^{m\pm}u^{n-}], \tag{32}$$

where the coefficients $C_{n\pm}^{m\pm}$ are taken from the asymptotics (14) and (15).

The system (31), (32) comprises 16 equations, and it expresses 16 derivatives of the unknown functions as linear combinations of these same functions. After some algebra this system can be rewritten in a compact matrix form (28).

Although the coordinate equations contain partial derivatives, the properties of the equations are closer to the properties of an ordinary differential equation. Namely, a particular solution of (28) is specified by a value at a single point, i.e., the space of solutions has dimension eight. To explain this, take an arbitrary point Q and fix the vector $\mathbf{u}$ at this point, then connect any other point Q' to Q by a line. Equations (28) can be restricted to the path joining Q' to Q, thus giving an ordinary differential equation with fixed initial conditions.

Note that the coefficients of the equations contain several constants, namely $C_{n\pm}^{m\pm}$, which depend only on k_0 and on the geometrical parameters a and b.

4.2 Compatibility and the Helmholtz property for the coordinate equations

As it is stated in the Frobenius theorem, local solvability of the system (28) follows from compatibility of the equations, i.e., from the identity

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\mathbf{u} = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\mathbf{u}. \tag{33}$$

In its turn, this follows from the identity for the coefficients of the coordinate equations:

$$\frac{\partial}{\partial z}\mathbf{Z}_- + \mathbf{Z}_-\mathbf{Z}_+ = \frac{\partial}{\partial \bar{z}}\mathbf{Z}_+ + \mathbf{Z}_+\mathbf{Z}_-. \tag{34}$$

Each component of the vector $\mathbf{u}$ must obey the Helmholtz equation, i.e.,

$$4\frac{\partial^2\mathbf{u}}{\partial z\partial\bar{z}} + k_0^2\mathbf{u} = 0. \tag{35}$$

This relation follows from the identity for the coefficients

$$4\left(\frac{\partial}{\partial z}\mathbf{Z}_- + \mathbf{Z}_-\mathbf{Z}_+\right) + k_0^2\mathbf{I}_8 = 0 \tag{36}$$

where $\mathbf{I}_8$ is the 8×8 identity matrix. Elementary algebraical analysis shows that relations (34), (36) are equivalent to

$$\mathbf{C}_+^-\mathbf{C}_-^+ = -\frac{k_0^2}{4}\mathbf{I}, \tag{37}$$

$$\hat{\mathbf{C}}_+^+\mathbf{C}_-^+ + \mathbf{C}_-^+\hat{\mathbf{C}}_-^- = 0. \tag{38}$$

Relations (37) and (38) are very important as they reduce the number of unknown parameters (degrees of freedom) of the coefficients of the coordinate equations. Furthermore, relations (37) and (38) are sufficient conditions for compatibility and the Helmholtz property. One can also prove that they are necessary conditions; this conclusion, however, requires further sophisticated analysis of the coordinate equations.

4.3 Identities following from reciprocity

Beside the relations (37) and (38), there are two other groups of relations reducing the number of free parameters of the coefficients. They are relations following from the geometrical symmetries (with respect to the coordinate axes) and relations following from the reciprocity theorem. The relations following from symmetries are quite obvious, for example,

$$C_{1+}^{2+} = -C_{3-}^{2-} = -C_{3+}^{4+} = -C_{1-}^{4-}.$$

Here we discuss less obvious relations following from the reciprocity principle.

The edge Green's functions are generated by the sources having configurations $S_{n\pm}$ (see (13)). The vertex asymptotic coefficients can be defined by a limit process similar to (24):

$$C_\beta^\alpha = \lim_{\epsilon\to 0}\frac{1}{8}\iint u^\alpha S_\beta \mathrm{d}s, \tag{39}$$

where α and β are indices composed of a number $m = 1\ldots 4$ and a sign. Due to the reciprocity theorem one can interchange the field and the source in (39). As a result, one obtains the relation

$$C_\beta^\alpha = C_\alpha^\beta, \tag{40}$$

which is valid for any α and β.

5 The spectral equation

A spectral equation is an ordinary differential equation for the directivities of the edge Green functions. This equation can be derived by studying certain Wronsky determinants [7]. Here, however, we prefer to derive the spectral equation from the coordinate equations.

Compose two vectors of unknowns, each of dimension 4:

$$\mathbf{U}^+(\varphi, N) = (U^{1+}, U^{2+}, U^{3+}, U^{4+})^T, \quad \mathbf{U}^-(\varphi, N) = (U^{1-}, U^{2-}, U^{3-}, U^{4-})^T,$$

and one vector of dimension 8:

$$\mathbf{U} = \begin{pmatrix}\mathbf{U}^+\\ \mathbf{U}^-\end{pmatrix}. \tag{41}$$

Note that the unknown elements in the above vectors are the directivities of the edge Green functions.

Theorem 4 *Vectors* $\mathbf{U}^+$ *and* $\mathbf{U}^-$ *are connected by the relation*

$$\mathbf{U}^- = \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi}}{k_0}\mathbf{C}_+^-\mathbf{U}^+. \tag{42}$$

Vector $\mathbf{U}^+$ *obeys an ordinary differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathbf{U}^+ = -\left(\frac{\mathrm{i}}{2}\mathbf{I} + \mathrm{i}\hat{\mathbf{C}}_+^+ + \frac{\mathrm{e}^{-\mathrm{i}\varphi}k_0}{2}\mathbf{P} + \frac{2\mathrm{e}^{\mathrm{i}\varphi}}{k_0}\mathbf{C}_-^+\bar{\mathbf{P}}\mathbf{C}_+^-\right)\mathbf{U}^+. \tag{43}$$

Equation (43) is the spectral equation for the problem. To prove this theorem we start with the system (28) and transform the derivatives with respect to the complex coordinates z and $\bar{z}$ into derivatives with respect to the polar coordinates φ and R:

$$\frac{\partial}{\partial R}\mathbf{u} = \frac{1}{\sqrt{z\bar{z}}}(z\mathbf{Z}_+ + \bar{z}\mathbf{Z}_-)\mathbf{u}, \tag{44}$$

$$\frac{\partial}{\partial\varphi}\mathbf{u} = \mathrm{i}(z\mathbf{Z}_+ - \bar{z}\mathbf{Z}_-)\mathbf{u}. \tag{45}$$

Now substitute the asymptotic form, (21), of the vector $\mathbf{u}$ for large R in (44). At leading order in the far field as $R \to \infty$, it is found that

$$\mathrm{i}k_0\mathbf{U} = -2\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}\mathbf{C}_-^+ \\ \mathrm{e}^{\mathrm{i}\varphi}\mathbf{C}_+^- & 0 \end{pmatrix}\mathbf{U}. \tag{46}$$

This relation is equivalent to (42). Now substitute (21) in (45). Again, from the leading-order terms in the far-field asymptotics, it is seen that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathbf{U} = -\mathrm{i}\begin{pmatrix} \mathbf{I}/2 + \hat{\mathbf{C}}_+^+ & \mathrm{e}^{-2\mathrm{i}\varphi}\mathbf{P}\mathbf{C}_-^+ - \mathbf{C}_-^+\bar{\mathbf{P}} \\ \mathbf{C}_+^-\mathbf{P}_+ - \mathrm{e}^{2\mathrm{i}\varphi}\bar{\mathbf{P}}\mathbf{C}_-^- & -(\mathbf{I}/2 + \hat{\mathbf{C}}_-^-) \end{pmatrix}\mathbf{U}. \tag{47}$$

On combining (47) with (42) and taking into account (37) an equation of order 4 for the vector we obtain $\mathbf{U}^+$, namely (43).

6 On finding the edge Green's functions directivities by using the spectral equation

Theorem 4 is the main result of the paper. This theorem states that the directivities of the edge Green's functions form a solution of an ordinary differential equation. The coefficients of the equation are elementary functions of the scattering angle, but they contain several unknown constants. The constants are the elements of the matrices $\mathbf{C}_-^+$, $\mathbf{C}_+^-$, $\hat{\mathbf{C}}_+^+$ and $\hat{\mathbf{C}}_-^-$. Moreover, the initial condition for the equation is unknown.

Yet, solutions of the spectral equations should obey certain restrictions. These restrictions follow from the fact that the components of the solutions are directivities of wave fields with specified branch points. The authors are planning to publish a separate paper describing, in detail, the restrictions that should be imposed on solutions of the spectral equation. Without going into details, let us mention here that a solution of the spectral equation can be used for reconstruction of a solution of the coordinate equations. The reconstruction is provided by an integral transformation. The behaviour of the reconstructed solution of the coordinate equations at possible branch points can be revealed by studying the asymptotics of the solution as $\mathfrak{Im}\,\varphi \to \pm\infty$. Thus, the restrictions have form of equations formulated for the coefficients of the asymptotics of the directivities. Using the language of ordinary differential equations, the restrictions are imposed on the extended monodromy data of the solution.

The restrictions are not obeyed by the solutions of the spectral equations having parameters $\mathbf{C}_\beta^\alpha$ arbitrarily chosen. Therefore the restrictions can be considered as implicit conditions imposed on the constant parameters of the spectral equation. It can be shown that the number of restrictions is exactly the same as the number of degrees of freedom for the constant parameters (i.e., the number of free parameters remaining after taking into account all algebraic links between them). The unknown constant parameters for the spectral equation can be calculated numerically as a result of, say, an optimization procedure. The target function for this procedure is a sum of discrepancies for all restrictions. Unfortunately, no analytical method for finding the unknown constant parameters can be proposed.

7 Conclusion

The problem of acoustic scattering at a gap between two orthogonal semi-infinite barriers has been reformulated as a propagation problem on a multi-sheet surface. An auxiliary propagation problem has been obtained by setting a single plane wave as the incidence field.

A set of eight edge Green's functions were introduced on the branched surface. Four theorems involving these functions have been proved. The first two theorems connect the wave field of the auxiliary problem with the edge Green's functions. The third theorem states that the vector composed of the edge Green's functions obeys the coordinate differential equations, the coefficients of which are rational functions of the coordinates. The fourth theorem states that the vector composed of the directivities of the edge Green's functions obeys an ordinary differential equation (the spectral equation). The coefficients of this equation are elementary functions of the scattering angle.

No numerical results have been presented since the main purpose of this paper was to propose a solution method for complicated scattering problems such as that described herein. The procedure is summarized below.

First, the unknown constants, i.e., the elements of the matrices $\mathbf{C}^{+}_{-}$, $\mathbf{C}^{-}_{+}$, $\hat{\mathbf{C}}^{+}_{+}$ and $\hat{\mathbf{C}}^{-}_{-}$ are found numerically for given k_0, a and b. These constants are chosen to satisfy a set of restrictions that guarantee the correct behaviour of the edge Green's functions on the branched surface. The structure of these restrictions and the optimization procedure for finding the unknown coefficients are the subject of another paper by the authors.

The spectral equation is then solved. A single solution is found among its four linearly independent solutions. The required solution must have certain periodicity properties. This solution corresponds to the directivities (41). Thus, the functions $U^{m\pm}(\varphi, N)$ become known.

The directivity $D(\psi, 1; \varphi, N)$ is obtained via formula (22), and finally, the directivity for the physical problem is reconstructed by symmetrization (6).

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